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BEND, BREAK AND COUNT

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ABSTRACT

This paper gives a formula for the number of members of a given 'nice' family of rational curves on a surface passing through the appropriate number of general points, expressing this number in terms of reducible members of the family. Similar formulae have been obtained previously using methods of quantum cohomology, but the present method is by contrast completely elementary, relying merely on some simple geometry on ruled surfaces.

The purpose of this paper is establish and apply an enumerative formula or 'method' dealing with a family $\mathcal{C} = \{\bar{C}_y : y \in Y\}$ of rational curves on a variety S, e.g., a rational surface. Significantly, the family $\mathcal{C}$ is not assumed to be the family of 'all' rational curves of given homology class: rather, we require only that it be sufficiently large ($n = \dim Y \geq 3$) and well-behaved as regards deformation theory and codimension-1 degenerations. The formula computes the 'degree' $d(\mathcal{C})$, i.e., the number of curves $\bar{C}_y$ through n general points of S, in terms of analogous degree-type numbers attached to the (codimension-1) boundary component's Z of Y, which parametrize the reducible curves (there are several such numbers depending on how the components of the reducible curve are 'weighted').

Some comments are in order on connections with quantum cohomology. While the author denies any first-hand knowledge or understanding of the latter, its algebro-geometric aspect has been represented as essentially equivalent to certain recursive formulae for counting rational curves, which are contained in the associativity formula for quantum multiplication. It has seemed to the author that these recursions, at least, are largely a matter of taking advantage of the

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'slack' in the problem, i.e. the large number of deformation parameters for rational curves (on 'convex' varieties). This viewpoint suggests a connection with Mori's bend-and-break technique, a small part of which is the observation that once a rational curve 'bends' sufficiently (on a surface, this means moving in a 3-parameter family) it will 'break', i.e., admit a reducible limit. Our general formula (Theorem 1, Section 1) is merely a quantitative version of this idea. As already indicated, it applies to any given (good enough) family and accordingly does not rely on existence of (compact) moduli spaces for (reparametrization classes of) stable maps as in [FP]. The proof is a completely elementary argument involving (multi-) sections and fibre components on a birationally ruled surface.

Now for better or worse, the effect of Theorem 1 is to shift the difficulty elsewhere, namely to 'enumerating' the families parametrized by the boundary components Z, which in principle is a lower-degree problem, but not necessarily well-behaved. The simplest Z are of 'product type' and parametrize a pair of independently varying curves plus a point of their intersection: these are unproblematic. However, there are others, such as those parametrizing a pair of mutually tangent curves, and worse: a variety of examples is given in Section 2.

In Section 3 we consider the problem of enumerating plane curves of given degree d and given moduli, i.e., birational to a fixed smooth curve C of genus g (the analogous problem with fixed d, g and unrestricted moduli, a.k.a. the degree of the Severi-variety having long been settled [R]) (it should be pointed out that the main recursion formula of [R] contains a misprint which has been corrected in [R1]). If $N(d,g\rangle$ denotes the number of such curves through $3d-2g+2$ general points (or $3d-1$ if $g=1$) then Pandharipande [P] has shown (where we use the standard notation $N_d = N(d,0\rangle$)

$$N(d,1\rangle = \frac{(d-1)(d-2)}{2} N_d.$$

Here we give a recursive procedure for computing $N(d,2\rangle$. While it is fairly clear what sub-problems would have to be solved to extend this to higher genus, it is unclear whether those can be solved, especially ones involving high-order contact between rational curves. However, see [R′] for a different approach to this problem. In the appendix we prove a proposition of independent interest about representing a divisor classs on a Del Pezzo surface by a rational curve.

The referee points out that closely related results have also been obtained by Caporaso and Harris (to appear in Compositio Mathematica); on the other hand, the method of this paper has been extended by the author to some other cases such as elliptics and cuspidals in the plane [R2] and rational space curves [R3].

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1. General formula

Suppose given a flat family $\{\bar{C}_y : y \in Y\}$ of curves on a smooth projective surface S, parametrized by an irreducible n-dimensional projective variety Y, such that a general $\bar{C}_y$ is a irreducible rational curve. Thus we have a diagram

$$(1.1)\qquad \begin{array}{l} \mathcal{C} \subset Y \times S \xrightarrow{f=p_2} S \\ \bar{\pi} \downarrow \\ Y \end{array}$$

where $\bar{\pi}$ is flat with fibres $\bar{\pi}^{-1}(y) = \bar{C}_y$. We assume Y is normal. In what follows, only geometry is codimension ≤ 1 on Y will play any role (so we could actually assume Y nonsingular without either losing generality or gaining convenience). Let $n: \mathcal{C} \to \bar{\mathcal{C}}$ be the normalization and $\pi = \bar{\pi}n$ the resulting flat family with fibres $C_y = \pi^{-1}(y)$ mostly isomorphic to $\mathbb{P}^1$. We let $\partial Y \subset Y$, the 'boundary', denote the discriminant locus of π. We now introduce a strong 'good behavior' condition on our family which, while not absolutely essential, makes for a simpler enumeration formula and is satisfied in applications.

(1) (*) Y is normal, hence $\mathcal{C}$ is smooth in codimension 2 except at singular points of fibres; for a generic point z of any $(n-1)$-dimensional component Z of ∂Y, the fibre C_z has just two components $C_{1,z}, C_{2,z}$, both $\mathbb{P}^1$'s, and $C_{1,Z} \cap C_{2,Z} = \{p\}$ is an $A_{\ell-1} \times \mathbb{C}^{n-1}$ singularity on $\mathcal{C}$ for some $\ell = \ell(Z)$.

Now the most basic numerical character attached to the family $\mathcal{C}$ is the **degree** (classically, the **grade**), which we denote by $d(\mathcal{C})$ (or sometimes, when $\mathcal{C}$ is understood, by $d(Y)$, and which is defined to be the number of $y \in Y$ such that $\bar{C}_y$ contains $n = \dim Y$ general points $s_1, \ldots, s_n \in S$, i.e.

$$d(\mathcal{C}) = \#\{y \in Y : \bar{C}_y \ni s_1, \ldots, s_n\}.$$

To define analogues of degree for boundary components, it will be convenient to count each one twice by introducing a 'marking'. Thus define a **marked** boundary component Z to consist of a boundary component Z' together with an ordering of the two corresponding components $\bar{C}_{1,z}, \bar{C}_{2,z}, z \in Z$; if the components are monodromy-interchangeable we still consider both orderings. To this data we associate the following degrees:

$$\begin{aligned} & d^{1,1}(Z) = \#\{z \in Z\colon \bar{C}_z \ni s_1, \dots, s_{n-1}, s_1 \in \bar{C}_{1,z},\ s_2 \in \bar{C}_{2,z}\}, \\ (1.2)\qquad & d^{0,2}(Z) = \#\{z \in Z\colon \bar{C}_z \ni s_1, \dots, s_{n-1}, s_1, s_2 \in \bar{C}_{2,z}. \end{aligned}$$

The simplest—though by no means all—boundary loci are those where the two components vary independently. More precisely, let us say that a boundary locus, sum of boundary components, is of **product type** if it can be naturally identified with a locus $\{(\bar{C}_1, \bar{C}_2,\ p \in \bar{C}_1 \cap \bar{C}_2)\}$, where $\bar{C}_i \in \{\bar{C}_i\}$ are independtly generic in their respective (irreducible) families of dimension $n_i,\ n_1 + n_2 = n - 1,\ \bar{C}_1$ and $\bar{C}_2$ meet transversely and $p \in \bar{C}_1 \cap \bar{C}_2$ can be specified arbitrarily (obviously in exactly $\bar{C}_1.\bar{C}_2$ ways). For Z of product type, clearly

$$\begin{aligned} & d^{1,1}(Z) = \binom{n-1}{n_1 - 1} d(\{\bar{C}_1\}) d(\{\bar{C}_2\}) \bar{C}_1.\bar{C}_2, \\ (1.3)\qquad & d^{0,2}(Z) = \binom{n-1}{n_1} d(\{C_1\}) d(\{\bar{C}_2\}) \bar{C}_1.\bar{C}_2. \end{aligned}$$

THEOREM 1: *In the above situation, suppose moreover $n \geq 3$. Then for any line bundle L on S, we have*

$$(1.4)\qquad L^2 d(Y) = \sum_Z \ell(Z)[(C_{1,z}.L)(C_{2,z}.L) d^{1,1}(Z) - (C_{1,z}.L)^2 d^{0,2}(Z)],$$

the sum being over all marked boundary component Z with corresponding generic curves $C_{1,z}, C_{2,z}$

Proof: First we might as well cut Y down to a 3-fold by imposing $s_4, \dots, s_n$ and henceforth assume $n = 3$. Then cut Y down to a (smooth) curve B by imposing s_1, s_2 and minimally resolve $C_Y \times B$, thus obtaining a diagram

$$(1.5)\qquad \begin{array}{ccc} X & \xrightarrow{f} & S \\ \pi \downarrow & & \\ B & & \end{array}$$

with X a smooth surface, π a blown-up $\mathbb{P}^1$-bundle with sections S_i corresponding to $s_i, i = 1, 2$, and reducible fibres of the form

$$(1.6)\qquad C^i = C_1^i + E_1^i + \cdots + E_{\ell_i - 1}^i + C_2^i$$

with $f(E_1^i+\cdots+E_{\ell-1}^i)=p^i$, a point on S, and $\ell_i=\ell(Z_i)$ where Z_i is the boundary component whence C^i comes. For future use, we note here that knowing the structure of the reducible fibres C^i is equivalent to knowing the singularity type of C along Z_i, and in any event is the only thing we need to know this singularity for. It is obvious—but nevertheless crucial—that the divisor class group of X is generated by any section plus fibre components. Note that $(E_j^i)^2=-2,(C_j^i)^2=-1$. It is easy to see from this firstly that

$$(1.7)\quad S_1-S_2\sim\sum_{\substack{s_1\in C_1^i\\ s_2\in C_2^i}}\left(\sum_{j=1}^{\ell_i-1}(jE_j^i+\ell_iC_2^i)+\sum_{\substack{s_2\in C_1^i\\ s_1\in C_2^i}}\left(\sum_{j=1}^{\ell_1-1}(\ell_i-j)E_j+\ell_iC_1^i\right)\right)-mF,$$

$F=$ fibre. Taking the dot product with S_1 yields $-m=S_1^2$.

As s_1 and s_2 are interchangeable by a suitable monodromy tranformation and dot products are preserved, we also have $-m=S_2^2$, so taking the dot product of (1.7) with S_2 yields

$$(1.8)\qquad m=\sum_{\substack{s_1\in C_1^i\\ s_2\in C_2^i}}\ell_i=\sum_Z\ell(Z)d^{1,1}(Z).$$

Now set $d=L.\bar{C}_y$ and note as before that dS_1-f^*L is a linear combination of fibre components, hence one can easily check that

$$(1.9)\qquad dS_1-f^*L\sim\sum_{\substack{s_1\in C_1^i\\ s_2\in C_1^i vC_2^i}}(C_2^i.L)\Big(\sum_{j=1}^{\ell_i-1}jE_j^i+\ell_iC_2^i\Big)$$
$$+\sum_{\substack{s_1\in C_2^i\\ s_2\in C_1^i\cup C_2^i}}(C_1^i.L)\Big(\sum(\ell_i-j)E_j^i+\ell_iC_1^i\Big)-aF.$$

Then taking the dot product with S_1 yields $a=dm$. Then squaring both sides we get (using the notation $C_{i,Z}.L$ to denote $C_{i,z}.L$ for generic z in the boundary component Z)

$$\begin{aligned}
L^2d(Y) = (f^*L)^2 = d^2m - &\sum_{\substack{s_1\in C_1^i \\ s_2\in C_1^i\cup C_2^i}} \ell_i(C_2^i.L)^2 - \sum_{\substack{s_1\in C_2^i \\ s_2\in C_1^i\cup C_2^i}} \ell_i(C_1^i.L)^2 \\
&= \sum_{\substack{s_1\in C_1^i \\ s_2\in C_2^i}} (C_1^i.L + C_2^i.L)^2\ell_i - \sum_{\substack{s_1\in C_1^i \\ s_2\in C_1^i\cup C_2^i}} \ell_i(C_2^i.L)^2 - \sum_{\substack{s_1\in C_2^i \\ s_2\in C_1^i\cup C_2^i}} \ell_i(C_1^i.L)^2 \\
&= 2\sum_{\substack{s_1\in C_1^i \\ s_2\in C_2^i}} (C_1^i.L)(C_2^i.L)\ell_i - \sum_{s_1,s_2\in C_1^i} \ell_i(C_2^i.L)^2 - \sum_{s_1,s_2\in C_2^i} \ell_i(C_1^i.L)^2 \\
&= \sum_Z d^{1,1}(Z)\ell(Z)(C_{1,z}.L)(C_{2,z}.L) - \sum d^{0,2}(Z)\ell(Z)(C_{1,z}.L)^2. \qquad \blacksquare
\end{aligned}$$

2. Various examples

(a) Del Pezzo surfaces: all curves. We begin by recalling some facts. Let S be a Del Pezzo surface, with (ample) anticanonical bundle $-K$ and Picard group N. Let us call a class $C \in N$ **good** if $-KC \geq 0$ and $C^2 \geq 0$. Then

(i) if C is the class of an integral curve $\bar{C}$, then either $\bar{C}$ is a line ($-K\bar{C} = 1$, $\bar{C}^2 = -1$), or C is good;

(ii) If $\bar{C}$ is an irreducible rational curve, then as such $\bar{C}$ has unobstructed deformations of dimension $-K.\bar{C} - 1$ which are generally transverse to given subvarieties. (See, for instance, [H] or Kollàr's book on rational curves.)

Now (i) and (ii) easily imply the following:

(iii) if $\bar{C} \subset S$ is a good rational curve and Y is the normalization of the locus $\{\bar{C}\}$ of rational curves in the linear system $|\bar{C}|$, then the hypotheses of Theorem 1 are satisfied provided $n = -K.\bar{C} - 1 \geq 3$; the boundary loci Z all have $\ell(Z) = 1$, are of product type and correspond to (ordered) expressions

$$c = [\bar{C}] = c_1 + c_2 \tag{2.a.1}$$

with c_1, c_2 representable by (irreducible) rational curves.

To see that (iii) holds consider the general divisor $D = \sum_1^r m_i\bar{C}_i$ on S corresponding to some boundary divisor Z of Y and say $r > 1$. Then D can a priori move in a multiplicity- and component-preserving family of dimension at most

$$\sum(-K.\bar{C}_i - 1) \leq (D.-K) - r \leq D.-K-2 = \dim(Z). \tag{$*$}$$

It follows that equality must hold throughout ($*$), so that $r = 2, m_1 = m_2 = 1$, and moreover $\bar{C}_1, \bar{C}_2$ are general in their respective families and in particular mutually transverse, which easily implies that $\ell(Z) = 1$ and Z is of product type. The case $r = 1$ easily implies that $\bar{C}_1$ is a line, contradicting goodness. This proves (iii).

Explicitly, taking $L = -K$ leads to the following, in which we denote $d(Y)$ by N_c:

$$K^2 N_c = \sum_{c=c_1+c_2} N_{c_1} N_{c_2} \Bigg[(-K.c_1)(-K.c_2)(c_1.c_2) \binom{-K.c-4}{-K.c_1-2} - (-K.c_1)^2 c_1.c_2 \binom{-K.c-4}{-K.c_1-1} \Bigg] \tag{2.a.2}$$

Now at least if S has anticanonical degree ≥ 3, then any good class C is representable by a rational curve (see Appendix). Whenever this is so, the problem of effectively computing the RHS of (2.a.2) becomes a purely combinatorial matter. For $S = \mathbb{P}^2$, (2.a.2) reduces to the 'associativity relation' of Kontsevich et al., cf. [FP].

(b) Higher dimensions. There are potentially several ways to meaningfully extend Theorem 1 to the case of a higher-dimensional ambient variety. Without getting systematically involved in this matter here, we shall merely indicate a relatively obvious such extension, obtained by simply replacing point conditions by incidence with respect to codimension-2 linear spaces; see also [R3]. Let $\{\bar{C}_y : y \in Y\}$ etc. be as in Section 1, except that $m = \dim S$ is no longer assumed to equal 2, while the line bundle L is assumed very ample. We may then define

$$d_L(Y) = \#\{y : \bar{C}_y \cap L_1^i \cap L_2^i \neq \emptyset \quad i = 1, \dots, n\}, \quad L_j^i \in |L| \quad \text{general}$$

and likewise for the $d_L^{i,j}(Z)$. The same arguments apply, notwithstanding that the analogues of S_1 and S_2 are now only multisections: the essential point is that, still, $S_1^2 = S_2^2$. The following formula then obtains

$$d_L(Y) = \sum \ell(Z)[(C_1L)(C_2L)d_L^{1,1}(Z) - (C_1.L)^2 d_L^{0,2}(Z)]. \tag{2.b.1}$$

(c) The plane: some codimension-1 counts. Here we give counts associated with some codimension-1 loci in the family of all rational plane curves, beginning with the number C_d of rational curves of degree d through $3d-2$ points having a node in a fixed line M. (The referee points out that a formula for C_d was also obtained by Caporaso–Harris and by Pandharipande). The marked boundary components (Z, C_1, C_2) are easily determined and come in two types, depending on whether C_1 and C_2 have a common point on M, or whether C_1 or C_2 has a node on M. For the first type we have, e.g.,

$$d^{1,1}(Z) = \left[\binom{3d-5}{3d_1-2} d_1 + \binom{3d-5}{3d_2-2} d_2\right] (d_1 d_2 - 1) N_{d_1} N_{d_2}, \tag{2.c.1}$$

while the second is of product type. Applying Theorem 1, we obtain a recursion:

$$\begin{aligned} C_d = \sum \Bigg[& \left(\binom{3d-5}{3d_1-2} d_2 - \binom{3d-5}{3d_1-1} d_1\right) d_1^2 \\ & + \left(\binom{3d-5}{3d_2-2} d_1 d_2 - \binom{3d-5}{3d_2-3} d_1^2\right) d_2 \Bigg] (d_1 d_2 - 1) N_{d_1} N_{d_2} \\ + \sum \Bigg[& \left(\binom{3d-5}{3d_1-3} + \binom{3d-5}{3d_2-3}\right) d_1 d_2 \\ & - \binom{3d^5}{3d_1-2} d_1^2 - \binom{3d-5}{3d_2-2} d_2^2 \Bigg] d_1 d_2 C_{d_1} N_{d_2}. \end{aligned} \tag{2.c.2}$$

Next, we consider some numbers which may be derived from C_d by elementary means. First, let B_d be the number of rational curves of degree d through $3d-2$ general points which are properly (i.e. at a smooth point) tangent to a fixed line M. Then B_d is related to C_d by the formula

$$B_d + 2C_d = 2(d-1)N_d. \tag{2.c.3}$$

This comes about by considering the (rational) 'restriction' map

$$\begin{aligned} r\colon V_{d,0} \subset \mathbb{P}^n & \cdots \to \mathbb{P}^d \\ \bar{C} & \mapsto \bar{C} \cap M, \end{aligned}$$

$$V_{d,0} = \text{variety of degree } -d \text{ rational curves},$$

which pulls back the discriminant hypersurface (of degree $2(d-1)$) to the sum of the properly tangent locus (with multiplicity 1) plus the 'node on M' locus (with multiplicity 2).

A natural generalization of B_d is the number $B_{d,e,g}$ of curves (rational degree -d, through $3d-2$ genral points) properly tangent to a given curve E of degree e and geometric genus g. To compute this we return to the situation of (1.5) where now $f\colon X \to \mathbb{P}^2$ has degree N_d, and note that $B_{d,e,g}$ coincides with the number of proper (smooth) ramification points of $\pi|_{f^{-1}(E)}$, and that the singularities of

E reduce the geometric genus of $f^{-1}(E)$ by $N_d((e-1)(e-2)/2 \ \ -g)$. Hence by the adjunction formula

$$\begin{aligned} B_{d,e,g} &= ef^*L(ef^*L+K_X-K_B)-2N_d((e-1)(e-2)/2 \ \ -g) \\ &= e(e-1)N_d+ef^*L(f^*L+K_X-K_B)-(e-1)(e-2)N_d+2gN_d \\ &= 2(e-1)N_d+eB_d+2gN_d \end{aligned} \tag{2.c.4}$$

where $L =$ line. In particular for $g = 0$ we obtain

$$B_{d,e,0} = 2(e-1)N_d + eB_d. \tag{2.c.5}$$

(d) SOME CROSS-RATIO COUNTS: Here we consider some (still codimension-1) counts involving a 'marked' rational curve $\bar{C}$. More precisely, we shall assume the normalization C of $\bar{C}$ is isomorphic to a $\mathbb{P}^1$ carrying an ordered quadruple of distinct points (p_1, q_1, p_2, q_2) such that the isomporphism class of

$$(\mathbb{P}^1, \{p_1, q_1\}, \{p_2, q_2\})$$

is fixed.

Note that

$$\mathrm{Aut}(\mathbb{P}^1, \{p_1, q_1\}\{p_2, q_2\}) = \mathbb{Z}_2$$

where a generator induces the permutation $(p_1, q_1)(p_2q_2)$ (if we identify $(p_1q_1) = (0, \infty)$ and view p_2, q_2 as complex numbers, this generator is given by the rational function $\frac{p_2q_2}{z}$).

Now fix reduced plane curves E_1, F_1, E_2, F_2 in general position of respective degrees e_i, f_j as well as a distinct quadruple $p_1, q_1, p_2, q_2 \in \mathbb{P}^1$. We consider the family Y of rational curves $\bar{C}$ of degree d which admit a parametrization $f \colon \mathbb{P}^1 \to \bar{C}$ such that

$$f(\{p_i, q_i\}) \subset E_i, \quad i = 1, 2.$$

This family is clearly $3d-2$-dimensional and, moreover, the number of curves in Y through $3d-2$ general points in the plane depends only on the degrees e_1, e_2.

We may therefore set

$$N(d < (e_1), (e_2)) := d(Y) = 1/2\#\{f\colon (\mathbb{P}^1, \{p_1, q_1\}, \{p_2, q_2\}) \to (\mathbb{P}^2, E_1, E_2)\}$$

(this is the number of f's up to source-isomorphism). Similarly let

$$N(d < e_1, f_1, (e_2)) = \#\{f\colon (\mathbb{P}^1, p_1, q_1, \{p_2, q_2\}) \to (\mathbb{P}^2, E_1, F_1, E_2)\}$$

(this time there are no source-automorphisms, hence no need for the 1/2 factor), and likewise $N(d < e_1, f_1, e_2, f_2)$. Specialization yields some easy relations among these numbers: for instance, by specializing a curve of degree $e_1 + f_1$ to one of the form $E_1 + F + 1$, we conclude

$$\begin{aligned} N(d < (e_1 + f_1), (e_2)) = & N(d < (e_1), (e_2)) + N(d < (f_1)(e_2)) \\ & + N(d < e_1, f_1, (e_2)) + N(d < f_1, e_1, (e_2)), \end{aligned}$$

etc.; also, these numbers possess an evident symmetry, e.g.,

$$N(d < e_1, f_1, (e_2)) = N(d < f_1, e_1, (e_2)).$$

It follows formally that

$$\begin{aligned} N(d < (e_1), (e_2)) = & e_1 e_2 N(d < (1), (1)) + (e_1 - 1)(e_2 - 1)(e_1 + e_2) N(d < 1, 1, (1)) \\ & + (e_1 - 1)(e_2 - 1) e_1 e_2 N(d < 1, 1, 1, 1), \end{aligned} \tag{2.d.1}$$

$$\begin{aligned} N(d < e_1, f_1, (e_2)) &= e_1 f_1 e_2 (N(d < 1, 1, (1)) + (e_2 - 1) N(d < 1, 1, 1, 1)), \\ N(d < e_1, f_2, e_2, f_2) &= e_1 f_1 e_2 f_2 N(d < 1, 1, 1, 1). \end{aligned}$$

It thus suffices to compute the basic numbers

$$N(d < (1), (1)), \quad N(d < 1, 1, (1)), \quad N(d < 1, 1, 1, 1),$$

for which we may set up a recursion in d based on Theorem 1. Consider, e.g., the case of $N(d < (1), (1))$. The curve $C = \mathbb{P}^1 \to \bar{C}$ of degree d will be marked with $\{p_1, q_1\}, \{p_2, q_2\}$, and the boundary components may be determined by an easy dimension count, e.g. based on the deformation theory of the moduli spaces of stable maps [FP] (though this is not essential). They correspond to pairs (C_1, C_2) where C_1 is of degree d marked with $\{p_1^0, q_1\}, \{p_2, q_2\}$ and isomorphic as such to $C_1, p_1^0 = C_1 \cap C_2$ and C_2 is of degree d_2 and marked with a 'new' p_1 playing the role of the old. It is clear that for such a component Z we have

$$\begin{aligned} d^{1,1}(Z) &= N(d_1 < 1, d_2, (1)) d_2 \binom{3d-5}{3d_2-2} + N_{d_1} N_{d_2} d_1^2 (d_1 - 1) \binom{3d-5}{3d_1-2} \\ &= N(d_1 < 1, 1, (1)) d_2^2 \binom{3d-5}{3d_2-2} + N_{d_1} N_{d_2} d_1^2 (d_1 - 1) \binom{3d-5}{3d_1-2}. \end{aligned} \tag{2.d.2}$$

Here the first summand comes from choosing $\bar{C}_2$ through $1+3d_2-2$ points, then a point on $\bar{C}_2 \cap E_1$, then $\bar{C}_1$; the second from choosing $\bar{C}_1$ through $1+3d_1-2$ points, then as an ordered pair on $\bar{C}_1 \cap E_2$, then a point $\bar{q}_1 \in \bar{C}_1 \cap E$—which in turn determines p_1^0 via cross-ratio—then finally $\bar{C}_2$. Applying Theorem 1 (recall that each Z has two markings so must be counted twice), we conclude:

$$\begin{aligned} N(d<(1),(1)) = \sum & \left[d_1 d_2^3 \binom{3d-5}{3d_2-2} - d_1^2 d_2^2 \binom{3d-5}{3d_1-1}\right] N(d_1<1,1,(1)) \\ & + \left[d_1^3 d_2 \binom{3d-5}{3d_1-2} - d_1^4 \binom{3d-5}{3d_2-1}\right] N(d_2<1,1,(1)) \\ & + \Bigg[d_1^3 d_2 (d_1-1) \binom{3d-5}{3d_1-2} - d_1^4 (d_1-1) \binom{3d-5}{3d_1-1} \\ & \quad + d_1 d_2^3 (d_2-1) \binom{3d-5}{3d_2-2} \\ & \quad - d_1^2 d_2^2 (d_2-1) \binom{3d-5}{3d_2-1}\Bigg] N_{d_1} N_{d_2}. \end{aligned} \tag{2.d.3}$$

Recursions for $N(d<1,1,(1))$ (involving $N(d<1,1,(1)), N(d<1,1,1,1)$ and N_d) and for $N(d<1,1,1,1)$ (involving $N(d<1,1,1,1)$ and N_d) may be obtained similarly.

For subsequent applications we require a 'dual' cross-ratio count when the marked curve $\bar{C}_1$ and $\bar{E}_1$ are fixed, as is the marking's cross-ratio, while $\bar{E}_2$ is allowed to vary (of course as a rational curve of given degree e_2 and through $3e_2-2$ general points). Thus define numbers

$$N(d,(e_1)>(e_2)) = \#\{f\colon (\mathbb{P}^1, \{p_1,q_1\}, \{p_2,q_2\}) \to (\bar{C}, \bar{E}_1, \bar{E}_2)\}$$

with the usual provisi, where $\bar{C}$ is fixed rational of degree d, E_1 fixed integral nodal of degree e_1, $\bar{E}_2$ rational of degree e_2 through $3e_2-2$ points, and the source of f is fixed up to isomorphism. We analogously define numbers

$$N(d,e_1,f_1>(e_2)), \quad N(d,(e_1),f_2>e_2), \quad N(d,e_1,f_1,f_2>e_2).$$

As before these behave simply with respect to the fixed unmarked curves, e.g.,

$$N(d,(e_1)>(e_2)) = e_1 N(d,(1)>(e_2)) + e_1(e_1-1)N(d,1,1>(e_2)). \tag{2.d.4}$$

Now to compute, e.g., $N(d,1,1,(e_2))$, the method of Theorem 1 yields a recursion in e_2. The boundary components $Z=\{E_{2,1}\cup E_{2,2}\}$ are easily determined and fall into two types depending on whether $p_2,q_2 \in E_{2,1}$, say (type (1)) or

$p_2 \in E_{2,1}$, $q_2 \in E_{2,2}$ (type (2)). The type (1) components are easily enumerated in terms of $N(d,1,1 > (e_{2,1}))$ and the type (2)'s in terms of

$$N(d,1,1,e_{2,1} > e_{2,2}) = e_{2,1}N(d,1,1,1 > e_{2,2}) = d^2 e_{2,1} N_{e_{2,2}}$$

(the latter equality is due to the fact that determining the position of p_1, q_1, p_2 determines that of q_2 via the cross-ratio). Thus one is finally reduced to computing, e.g., $N(d,1,1 > (1))$. Let's identify $(p_1, q_1) = (0, \infty)$, which then identifies cross-ratio with ordinary ratio. A moment's reflection shows that $N(d,1,1 > (1)) = d^2 M_d$, where M_d is the number of pairs $(a,b) \in \mathbb{C} \times \mathbb{C}$ such that $a/b \in \{\lambda - 1, \lambda^{-1}\}$ for a fixed general $\lambda \in \mathbb{C}$ and for a general degree-d map $f\colon \mathbb{P}^1 \to \mathbb{P}^1$, $f(a) = f(b)$. Specializing $\lambda \to 1$ keeping track of multiplicities and using the Riemann–Hurwitz formula, it is elementary that $M_d = 4(d-1)$, so

$$N(d,1,1 > (1)) = 4(d-1)d^2. \tag{2.d.5}$$

Similarly,

$$N(d,(1) > (1)) = 2(d-1)d^2 \tag{2.d.6}$$

(here one divides by 2 due to the involution $(p_1, q_1)(p_2 q_2)$). Thus one can compute all the above-mentioned cross-ratio counts.

3. Higher genus

For plane curves of positive genus, there are (at least) two types of counts one may wish to carry out, depending on whether the moduli of the curve are fixed or unrestricted. For unrestricted moduli one has the number $N(d,g)$ of (integral) curves of degree d and geometric genus g through $3d+g-1$ general points, for which a recursive formula was given in our earlier paper [R]. For fixed moduli, one has the number $N(d,g\rangle$ of integral curves of degree d birational to a fixed general smooth curve of genus g and passing through $3d-2g+2$ general points if $g \geq 2$ (or $3d-1$ points if $g=1$). The case $g=1$ was done by Pandharipande [P], who shows

$$N(d,1\rangle = \frac{(d-1)(d-e)}{2} N(d,0). \tag{3.1}$$

We now show how the method of this paper yields a procedure for computing $N(d,2\rangle$; see [R′] for a different approach to $N(d,g\rangle$ in general. (After this was first written the author became aware of an eprint by Katz, Qin and Ruan [KQR] which also considers $N(d,2)$ albeit from a slightly different viewpoint, based on Kleiman's triple-point formula; in view of this we'll be sketchy on some of the more tedious details.)

Now specializing an abstract curve C of genus 2 to a general binodal rational curve C_0, with normalization $(\mathbb{P}^1, \{p_1, q_1\}, \{p_2, q_2\}) \to (C_0, \text{node}, \text{node})$, we see that $N(d,2)$ may be identified with the number of maps

$$f\colon \mathbb{P}^1 \to \mathbb{P}^2$$

with image $\bar{C} = f(\mathbb{P}^1)$ passing through $3d-2$ general points, such that, for a fixed quadruple $(p_1, q_1, p_2, q_2), f(p_i) = f(q_i), i = 1, 2$, up to identifying $f \sim f \leq \alpha$ where α is the unique projective automorphism inducing the permutation $(p_1, q_1)(p_2, q_2)$ (i.e. the map induced by the limit of the hyperelliptic involution on C). To this Theorem 1 is applicable, and it remains to list the boundary components Z. These come in two types:

(1) $Z^1_{d_1, d_2}$, which corresponds to maps

$$(\mathbb{P}^1_1, \{p^0_1, q_1\}, \{p_2, q_2\}) \underset{p^0_1}{\cup} (\mathbb{P}^1_2, p_1, p^0_1) \xrightarrow{(f_1, f_2)} \mathbb{P}^2$$

where $\operatorname{im} f_i = \bar{C}_i$ has degree $d_i, f_1(p^0_1) = f_2(p^0_1), f_1(q_1) = f_2(p_1), f_1(p_2) = f_1(q_2)$. To enumerate $Z^1_{d_1, d_2}$ we introduce nodal cross-ratio counts

$$N(d < (e)) = \#\{f\colon (\mathbb{P}^1, \{p_1, q_1\}, \{p_2, q_2\}) \to (\mathbb{P}^2, E, \text{point})\}$$

where E is a fixed curve of degree e, the point is unspecified (i.e., the condition is $f(p_2) = f(q_2)$) and $\operatorname{im}(f)$ is a degree-d curve through $3d-2$ general points; similarly $N(d > (e))$ where $\operatorname{im} f$ (and the cross-ratio) are fixed and E is rational of degree e through $3e-2$ general points. With these we have, e.g.,

(3.2)
$$d^{1,1}(Z^1_{d_1,d_2}) = \binom{3d-2}{3d_2-2} N_{d_2} N(d_1 < (d_2)) + \binom{3d-2}{3d_1-2} N_{d_1} N(d_1 > (d_2)),$$

(3.3)
$$d^{0,2}(Z^1_{d_1,d_2}) = \binom{3d-2}{3d_2-3} N_{d_2} N(d_1 < (d_2)) + \binom{3d-2}{3d_1-1} N_{d_1} N(d_1 > (d_2)).$$

The numbers $N(d < (e))$ and $N(d > (e))$ may be computed recursively in analogy—as well as linkage with the cross-ratio numbers of Section 2(d): e.g. the recursion for $N(d < (e))$ involves $N(d_1 < (e))$ when p_2, q_2 go to a node on one component, as well as $N(d_1 < (d_2), (e))$ where the $\mathbb{P}^1$ splits so p_1 goes off to another component; and there will also be a $N(d_1,(e) > (d_2))$ where an unmarked curve of degree d_2 varies. Details are similar to the above.

(2) In this type

$$Z^2_{d_1,d_2} = \left\{ (\mathbb{P}^1_0, \{p^0_1, q^0_1\}, \{p_2, q_2\}) \underset{p^0_1}{\cup} (\mathbb{P}^1_1, p_1, p^0_1) \underset{q_1}{\cup} (\mathbb{P}^1_2, q_1, q^0_1)^{(f_0,f_1,f_2)} \longrightarrow \mathbb{P}^2 \right\},$$

where $\bar{C}_i = f_i(\mathbb{P}^1_i)$ have degrees $d_0 = 0, d_1, d_2, \bar{C}_1$ and $\bar{C}_2$ are *tangent* at $\bar{C}_0 = f_1(p^0_1) = f_2(q^0_1)$ and meet at $f_1(p_1) = f_2(q_1)$. This component has $\ell = 2$, and may be easily enumerated as in Section 2, e.g.,

$$\begin{aligned} d^{1,1}(Z^2_{d_1,d_2}) =& B_{d_1,d_2,0} \binom{3d-2}{3d_1-2} .(d_1d_2 - 2) + B_{d_2,d_1,0} \binom{3d-2}{3d_1-2} (d_1d_2-2) \\ =& (2(d_2-1)N_{d_1} + d_2B_{d_1}) \binom{3d-2}{3d_2-2} (d_1d_2-2) \\ & + (2(d_1-1)N_{d_2} + d_1B_{d_2}) \binom{3d-2}{3d_1-2} (d_1d_2-2), \end{aligned} \tag{3.4}$$

$$\begin{aligned} d^{0,2}(Z^2_{d_1,d_2}) =& (2(d_2-1)N_{d_1} + d_2B_{d_1}) \binom{3d-2}{3d_2-3} (d_1d_2-2) \\ & + (2(d_1-1)N_{d_2} + d_1B_{d_2}) \binom{3d-2}{3d_1-1} (d_1d_2-2). \end{aligned} \tag{3.5}$$

In this way we may obtain recursions for all the nodal cross-ratio numbers and hence compute $N(d,2\rangle$ via

$$N(d,2) = \sum d_1d_2(d^{1,1}(Z^1_{d_1,d_2}) + 2d^{1,1}(Z^2_{d_1,d_2})) - d_1^2(d^{0,2}(Z_{d_1,d_2}) + 2d^{0,2}(Z^2_{d_1,d_2})).$$

Appendix: Rational curves on Del Pezzo surfaces

The following result, though quite natural, seems to our knowledge to have escaped explicit mention in the literature.

PROPOSITION A: *Let S be a Del Pezzo surface of anticanonical degree at least 3 and c a divisor class on S with $c^2 > 0 > c.K$. Then c contains an irreducible rational curve.*

Proof: S is either a blown-up plane or a quadric, and we assume the former case as the latter is much simpler. By Riemann-Roch, c is clearly effective. If the (anticanonical) degree $-K.c = 1, 2$, our claim is easy, and likewise if $c = -K$. In general, we use an induction on $-K.c \geq 3$. Let D be a line on S (i.e.,

$D^2 = -1 = D.K$) such that $c.D$ is minimal, and set $c' = c - D$. If $c.D < 0$, induction clearly applies to c' yielding an irreducible rational representative C' moving in a family of rational curves of dimension $-K.c - 2 \geq 1$, and C' can therefore be specialized to (an image of) a connected rational chain C_0 meeting D. Then by easy and standard deformation theory (using ampleness of $-K$), $C_0 + D \simeq c$ can be deformed to an irreducible rational curve. If $c.D = 0$, look instead for the smallest strictly positive $c.D', D' =$ line and argue as below. (Or alternatively, blow D down and restart the argument—which basically amounts to the same thing.) Now let's assume $c.D > 0$. It will suffice to prove

$$(c - D)^2 > 0,$$

for then deformation theory can be applied as above to conclude. To this end let us consider a suitable model of S as a blowup of the plane in r points, $r \leq 6$, and represent c in the usual way as

$$(b; a_1 \geq \cdots \geq a_r),$$

where we may assume

$$a_r = c.E_r = c.D$$

is minimal as above, which translates into

$$b \geq a_1 + a_2 + a_r,$$

$$2b \geq a_1 + \cdots + a_5 + a_r,$$

and our claim amounts to

$$N(c) := b^2 - a_1^2 - \cdots - a_r^2 > 2a_r + 1.$$

Let's do some tail-flattening on the sequence $a_1, \ldots, a_r$. Now if $a_r < a_{r-1} < a_2$, say, we may perform a 'switching transformation' with

$$a'_{r-1} = a_{r-1} - 1, a'_{r-2} = a_{r-2} + 1$$

which will only decrease $N(c)$ without affecting a_r. Continuing in this way we will eventually achieve

$$a_3 = \cdots = a_r$$

and in the process the value of N has decreased, a_1 has increased while b, a_r have remained unchanged. Then doing a similar 'switch' with a_2 and a_1 we may further assume $a_2 = a_r$. Now suppose $a_1 \leq 2a_r - 1$. Clearly $a_r \leq 1/3b$, so

$$b^2 - a_1^2 - \cdots - a_r^2 \geq \frac{6-r}{9}b^2 + 4a_r - 1,$$

therefore $b^2 - a_1^2 - \cdots - d_r^2 > 2a_r + 1$ unless $a_r = 1, b = 3$ in which case our original class c must have coincided with the anticanonical class $c = -K = (3; 1, \cdots 1)$ which we assumed was not the case. Now if on the other hand $a_1 \geq 2a_r$, we have

$$b^2 \geq (a_1 + 2a_r)^2 \geq a_1^2 + 12a_r^2 > a_1^2 + (r-1)a_r^2 + 2a_r + 1$$

as $r \leq 6$. ∎

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